MAE 113.
Assignment 9.

§5.5 - Differentiation of Logarithmic Functions.

1. $f(x) = 5 \ln x$.

$f'(x) = \dfrac{5}{x}$.

3. $f(x) = \ln(x+1)$

$f'(x) = \dfrac{1}{x+1}$.

5. $f(x) = \ln x^{8}$ Rule ③ p. 372.

$= 8 \ln x$.

$f'(x) = \dfrac{8}{x}$.

7. $f(x) = \ln \sqrt{x}$

$= \ln x^{1/2}$

$= \frac{1}{2} \ln x$

$f'(x) = \frac{1}{2} \cdot \frac{1}{x} = \frac{1}{2x}$.

12. $f(x) = \ln(\overbrace{3x^2 - 2x + 1}^{g(x)})$ $\qquad \frac{d}{dx} \ln[g(x)] = \frac{1}{g(x)} \cdot g'(x)$.

$= \dfrac{1}{3x^2-2x+1} \cdot (6x-2)$

$= \dfrac{6x-2}{3x^2-2x+1}$.

13. $f(x) = \ln \dfrac{2x}{x+1} = \ln 2x - \ln(x+1)$ Rule ② p. 372

$f'(x) = \dfrac{1}{\cancel{2}x} \cdot \cancel{2} - \dfrac{1}{x+1} = \dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{1}{x}\,\dfrac{x+1}{x+1} - \dfrac{1}{x+1}\,\dfrac{x}{x}$

$= \dfrac{\cancel{x}+1-\cancel{x}}{x(x+1)} = \dfrac{1}{x(x+1)}$

15. $f(x) = x^2 \ln x$. Product rule. $f'g + fg'$

$f'(x) = \frac{d}{dx}[x^2] \cdot \ln x + x^2 \frac{d}{dx}[\ln x]$

$f'(x) = 2x \ln x + x^2\left(\frac{1}{x}\right)$

$= 2x \ln x + x$.

$= x(2\ln x + 1)$ ✓

21. $f(x) = \sqrt{\ln x}$. Use Chain Rule.

$= (\ln x)^{1/2}$ outside function

inside function.

$f'(x) = \underbrace{\frac{1}{2}(\ln x)^{-\frac{1}{2}}}_{\text{derivative of the outside function}} \underbrace{\frac{1}{x}}_{\text{derivative of the inside function.}}$

$= \dfrac{1}{2 \cdot x \sqrt{\ln x}}$

23. $f(x) = (\ln x)^3$

$f'(x) = 3(\ln x)^2 \frac{1}{x}$

$= \dfrac{3(\ln x)^2}{x}$.

29. $f(t) = e^{2t} \ln(t+1)$

$$f'(t) = \frac{d}{dt}\left[e^{2t}\right] \ln(t+1) + e^{2t} \frac{d}{dt}\left[\ln(t+1)\right]$$

$$= e^{2t} \cdot 2 \cdot \ln(t+1) + e^{2t} \frac{1}{t+1}(1).$$

$$= 2e^{2t} \ln(t+1) + \frac{e^{2t}}{t+1}$$

$$= e^{2t}\left[2\ln(t+1) + \frac{1}{t+1}\right].$$

$$= e^{2t}\left[2\ln(t+1)\frac{(t+1)}{t+1} + \frac{1}{t+1}\right]$$

$$= e^{2t}\left[\frac{2(t+1)\ln(t+1) + 1}{t+1}\right].$$

39. $y = (x-1)^2(x+1)^3(x+3)^4$

$\ln y = \ln[(x-1)^2(x+1)^3(x+3)^4]$

$\ln y = \ln(x-1)^2 + \ln(x+1)^3 + \ln(x+3)^4$

differentiate

$\ln y = 2\ln(x-1) + 3\ln(x+1) + 4\ln(x+3)$

$$\frac{y'}{y} = \frac{2}{x-1} + \frac{3}{x+1} + \frac{4}{x+3}$$

$$y' = y\left[\frac{2}{x-1} + \frac{3}{x+1} + \frac{4}{x+3}\right]$$

$$y' = (x-1)^2(x+1)^3(x+3)^4\left[\frac{2}{x-1} + \frac{3}{x+1} + \frac{4}{x+3}\right]$$

To get the same answer at the back of the book.

$$\frac{2}{(x-1)}\frac{(x+1)}{(x+1)}\frac{(x+3)}{(x+3)} + \frac{3}{(x+1)}\frac{(x-1)}{(x-1)}\frac{(x+3)}{(x+3)} + \frac{4}{x+3}\frac{(x-1)}{(x-1)}\frac{(x+1)}{(x+1)}$$

$$= \frac{2(x^2+4x+3) + 3(x^2+2x-3) + 4(x^2-1)}{(x-1)(x+1)(x+3)}$$

$$= \frac{2x^2+8x+6+3x^2+6x-9+4x^2-4}{(x-1)(x+1)(x+3)}$$

$$= \frac{9x^2+14x-7}{(x-1)(x+1)(x+3)}$$

Then $y' = (x-1)^2(x+1)^3(x+3)^4 \dfrac{9x^2+14x-7}{(x-1)(x+1)(x+3)}$

$$y' = (x-1)(x+1)^2(x+3)^3(9x^2+14x-7)$$

41. $y = \dfrac{(2x^2-1)^5}{\sqrt{x+1}}$

$(x+1)^{\frac{1}{2}}$

$\ln y = \ln(2x^2-1)^5 - \ln\sqrt{x+1}$

$\ln y = 5\ln(2x^2-1) - \frac{1}{2}\ln(x+1)$.

$\dfrac{y'}{y} = 5\,\dfrac{1}{2x^2-1}(4x) - \dfrac{1}{2}\,\dfrac{1}{x+1}(1)$

$\dfrac{y'}{y} = \dfrac{20x}{2x^2-1} - \dfrac{1}{2(x+1)}$

$y' = y\left[\dfrac{20x}{2x^2-1} - \dfrac{1}{2(x+1)}\right]$

$y' = \dfrac{(2x^2-1)^5}{\sqrt{x+1}}\left[\dfrac{20x}{2x^2-1} - \dfrac{1}{2(x+1)}\right]$

To get the answer at the back of the book.

$\dfrac{20x}{2x^2-1}\cdot\dfrac{2(x+1)}{2(x+1)} - \dfrac{1}{2(x+1)}\cdot\dfrac{2x^2-1}{2x^2-1}$

$= \dfrac{40x(x+1) - (2x^2-1)}{2(x+1)(2x^2-1)}$

So,

$$y' = \frac{(2x^2-1)^5}{\sqrt{x+1}}\left[\frac{40x(x+1)-(2x^2-1)}{2(x+1)(2x^2-1)}\right]$$

$$y' = \frac{(2x^2-1)^4}{2(x+1)^{3/2}}\left[40x(x+1)-2x^2+1\right]$$

$$y' = \frac{(2x^2-1)^4}{2(x+1)^{3/2}}\left[40x^2+40x-2x^2+1\right]$$

$$y' = \frac{(2x^2-1)^4}{2(x+1)^{3/2}}\left[38x^2+40x+1\right]$$

$$y' = \frac{(2x^2-1)^4(38x^2+40x+1)}{2(x+1)^{3/2}}$$

43. $y = 3^x$

$\ln y = \ln 3^x$

$\ln y = x \ln 3$

$\frac{y'}{y} = 1 \ln 3 + x(0)$

$\frac{y'}{y} = \ln 3$

$y' = y \ln 3$

$y' = 3^x \ln 3$

46. $y = x^{\ln x}$

$\ln y = \ln x^{\ln x}$

$\ln y = \ln x \ln x$

$\ln y = (\ln x)^2$

$\frac{y'}{y} = 2(\ln x) \frac{1}{x}$

$y' = \frac{y \, 2 \ln x}{x}$

$y' = \frac{x^{\ln x} \, 2 \ln x}{x}$

$y' = x^{\ln x} x^{-1} \, 2 \ln x$

$y' = x^{\ln(x) - 1} \, 2 \ln x$

$y' = 2 x^{\ln(x) - 1} \ln x$

49. $f(x) = \ln x^2$

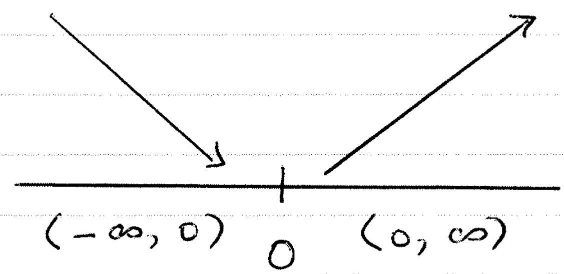
$f(x) = 2 \ln x.$

$f'(x) = \dfrac{2}{x}.$

$0 = \dfrac{2}{x}.$

$\dfrac{x}{2} = 0$

$\boxed{x = 0}$



$f'(x)$ $-$

$f'(-1) = \dfrac{2}{-1} = -2 < 0$ $\quad f'(1) = \dfrac{2}{1} = 2 > 0$

So f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

51. $f(x) = x^2 + \ln x^2$

$f(x) = x^2 + 2\ln x.$

$f'(x) = 2x + \dfrac{2}{x} = 2x + 2x^{-1}$

$f''(x) = 2 - 2x^{-2}$

$= 2 - \dfrac{2}{x^2}$

Set $f''(x) = 0.$

$0 = 2 - \dfrac{2}{x^2}$

$-2 = -\dfrac{2}{x^2}$

$\dfrac{2}{1} = \dfrac{2}{x^2}$

$x^2 = \dfrac{2}{2}$

$x^2 = 1$

$x = \pm\sqrt{1}$

$x = \pm 1.$

Another critical number is $x=0$ because $f''(x)$ does not exist.

$f(2) = 2 - \dfrac{2}{2^2}$

$= 2 - \dfrac{2}{4} = 2 - \dfrac{1}{2} = \dfrac{3}{2} > 0$

$f''(\tfrac{1}{2}) = 2 - \dfrac{2}{(\frac{1}{2})^2}$

$= 2 - \dfrac{2}{\frac{1}{4}}$

$= 2 - 2(4)$

$= 2 - 8 = -6 < 0$

$(-\infty, -1)$		$(-1, 0)$		$(0, 1)$		$(1, \infty)$
$\boxed{+}$	-1	$\boxed{-}$	0	$\boxed{-}$	1	$\boxed{+}$

$f''(x)$

$f''(-2) = 2 - \dfrac{2}{(-2)^2}$

$= 2 - \dfrac{2}{4}$

$= 2 - \dfrac{1}{2}$

$= \dfrac{3}{2} > 0.$

$f''(-\tfrac{1}{2}) = 2 - \dfrac{2}{(\frac{1}{2})^2}$

$= 2 - \dfrac{2}{(\frac{1}{4})}$

$= 2 - 2(4)$

$= 2 - 8$

$= -6 < 0.$

So f is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 0) \cup (0, 1)$.

§5.6 – Exponential Functions as Mathematical Models.

2. exponential decay $Q(t) = 2000e^{-0.06t}$ ← t is measured in years.

a) decay constant $k = 0.06$

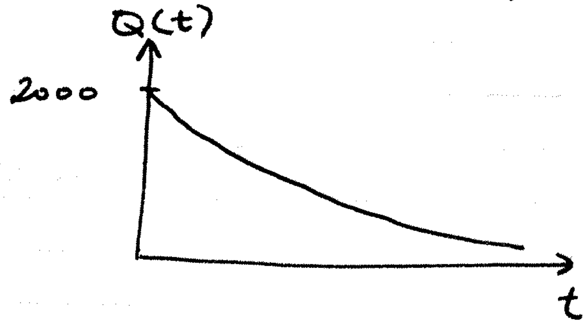
b) initial quantity $Q_0 = 2000$.

<u>or</u> $Q(0) = 2000e^{-0.06(0)} = 2000e^{0} = 2000(1) = 2000.$

c)

t	0	5	10	20	100
Q	2000	1481.636	1097.623	602.388	4.958

I used a calculator to calculate those values. But notice how it drops drastically.



3. E. coli doubles every 20 mins.

a) $Q(0) = 100$
$Q(20) = 200$
$Q(40) = 400$.

$\rightarrow$ $Q(t) = 100e^{kt}$

Now to find k using $Q(20)$.

$$\frac{200}{100} = \frac{100e^{k(20)}}{100}$$

$$2 = e^{k(20)}$$

Now solve for k.

Take ln of both sides

$$\ln 2 = \ln e^{k(20)}$$

$$\ln 2 = k(20)\underbrace{\ln e}_{1}$$

6. Resale value. follows exponential decay.

$$Q(0) = 500\,000 \longrightarrow Q(t) = 500\,000\, e^{-kt}$$
$$Q(3) = 320\,000 \longrightarrow 320\,000 = 500\,000\, e^{-3k}$$
$$Q(7) = ?$$

$$320\cancel{000} = 500\cancel{000}\, e^{-3k}$$
$$\frac{32}{50} = e^{-3k}$$

$$\ln\left(\frac{32}{50}\right) = \ln e^{-3k}$$

$$\ln\left(\frac{32}{50}\right) = -3k \underbrace{\ln e}_{1}$$

$$k = \frac{\ln(32) - \ln(50)}{-3}.$$

$$k \approx 0.148762$$

So the value is $Q(t) = 500000\, e^{-0.148762t}$ at time t.

$$\therefore\ Q(7) = 500\,000\, e^{-0.148762(7)}$$
$$= 176491.74$$ by calculator

So the machine's resale value will be \$176 491.74 4 years from now.

t in weeks.

13. $Q(t) = 120\left(1 - e^{-0.05t}\right) + 60 \qquad t \in [0, 20]$

the number of words per minute

a) $Q_0 = Q(0) = 120\left(1 - e^{-0.05(0)}\right) + 60$

$= 120(1 - \underbrace{e^{0}}_{1}) + 60$

$= 120(1 - 1) + 60$

$= 120(0) + 60$

$= 60.$

b) Halfway through the course is $t = 10$ weeks.

$Q(10) = 120\left(1 - e^{-0.05(10)}\right) + 60$

$= 120\left(1 - e^{-0.5}\right) + 60$

$= 120 - 120e^{-0.5} + 60$

$= 180 - 120e^{-0.5}$

≈ 107.21 words per minute

c) $Q(20) = 120\left(1 - e^{-0.05(20)}\right) + 60$

$= 120(1 - e^{-1}) + 60$

$= 120\left(1 - \frac{1}{e}\right) + 60$

$= 120 - \frac{120}{e} + 60$

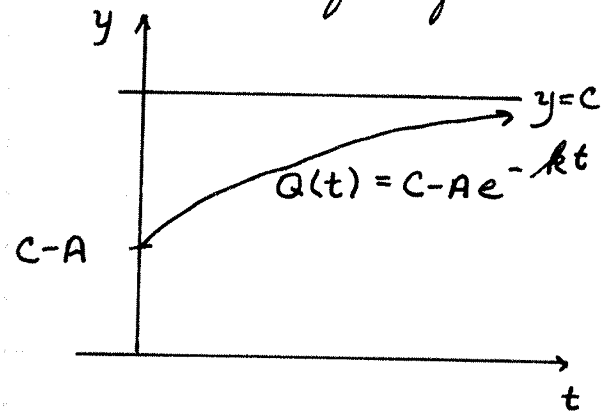
$= 180 - \frac{120}{e}$

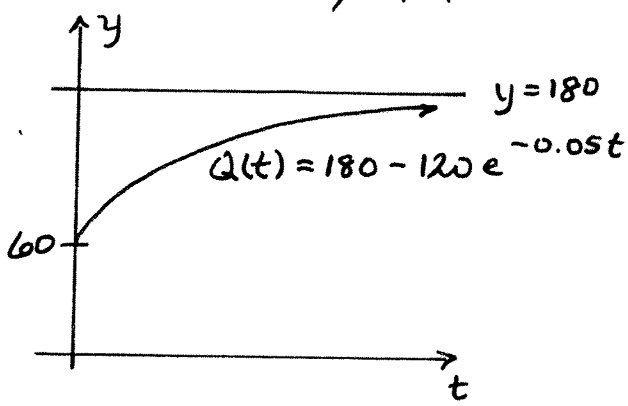
≈ 135.85 words per minute

To sketch the graph

$$Q(t) = 120(1 - e^{-0.05t}) + 60$$
$$= 120 - 120e^{-0.05t} + 60$$
$$= 180 - 120e^{-0.05t}$$

This now has the form of the learning curve as seen on p. 417.





17. $Q(t) = \dfrac{1000}{1+199e^{-0.8t}}$

a) $Q(1) = \dfrac{1000}{1+199e^{-0.8(1)}} = \dfrac{1000}{1+199e^{-0.8}}$

$= 11.05$ children

by calculator.

b) $Q(10) = \dfrac{1000}{1+199e^{-0.8(10)}} = \dfrac{1000}{1+199e^{-8}} = 937.42$ children

c) Let $t \to \infty$, then $e^{-0.8t} \to 0$

then $\lim\limits_{t\to\infty} Q(t) = \dfrac{1000}{1+0} = 1000$ children.

21. Spread of a Rumour.

$$f(t) = \frac{3000}{1 + Be^{-kt}}$$

$f(0) = 300.$

$f(2) = 600.$

$$300 = \frac{3000}{1+Be^{-k(0)}}$$

$$300 = \frac{3000}{1 + B\underbrace{e^{0}}_{1}}$$

Solve for b

$$300 = \frac{3000}{1 + B}$$

$$300(1+B) = 3000$$

$$300 + 300B = 3000$$

$$300B = 3000 - 300$$

$$300B = 2700$$

$$B = \frac{2700}{300}$$

$$B = 9$$

Solve for k

$$600 = \frac{3000}{1+9e^{-k(2)}}$$

$$600 = \frac{3000}{1+9e^{-2k}}$$

$$600(1+9e^{-2k}) = 3000$$

$$600 + 5400e^{-2k} = 3000$$

$$5400e^{-2k} = 3000 - 600$$

$$e^{-2k} = \frac{2400}{5400}$$

$$e^{-2k} = \frac{24}{54}$$

$$\ln e^{-2k} = \ln\left(\frac{24}{54}\right)$$

$$-2k \underbrace{\ln e}_{1} = \ln\left(\frac{24}{54}\right)$$

$$-2k = \ln\left(\frac{24}{54}\right)$$

$$k = \frac{\ln\left(\frac{24}{54}\right)}{-2}$$

$$k = 0.4055$$ by calculator.

Then $f(t) = \dfrac{3000}{1+9e^{-0.4055t}}$

$f(4) = \dfrac{3000}{1+9e^{-0.4055(4)}} = 1080.0965$ students heard about the policy after 4 hours.

§6.1 – The Antiderivative and Rules of Integration.

3. Verify just means to $\frac{d}{dx}F(x)$ to see that it matches $f(x)$.

$$\frac{d}{dx}\sqrt{2x^2-1} = \frac{d}{dx}(2x^2-1)^{\frac{1}{2}}$$

$$= \frac{1}{\cancel{2}}(2x^2-1)^{-\frac{1}{2}}(\overset{2}{\cancel{4}}x)$$

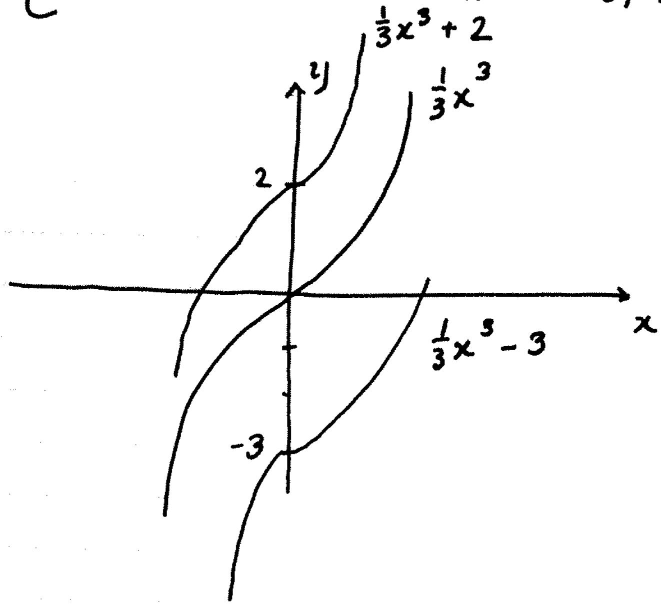
$$= \frac{2x}{\sqrt{2x^2-1}}$$

$$= f(x) \checkmark$$

7. a) $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2 = f(x) \checkmark$

b) $\frac{1}{3}x^3 + C$ remember the constant of integration.

c)



9. $\int 6\,dx = 6\int dx = 6x + C$ ✓

11. $\int x^3\,dx = \frac{1}{3+1}x^{3+1} + C = \frac{1}{4}x^4 + C$ ✓

13. $\int x^{-4}\,dx = \frac{1}{-4+1}x^{-4+1} + C = -\frac{1}{3}x^{-3} + C = -\frac{1}{3x^3} + C$ ✓

19. $\int \frac{2}{x^2}\,dx = 2\int \frac{1}{x^2}\,dx = 2\int x^{-2}\,dx$

$$= 2\cdot\frac{1}{-2+1}x^{-2+1} + C$$

$$= 2\cdot(-1)x^{-1} + C$$

$$= \frac{-2}{x} + C.$$ ✓

21. $\int \pi\sqrt{t}\,dt = \pi\int\sqrt{t}\,dt = \pi\int t^{1/2}\,dt$

$$= \pi\frac{1}{\frac{1}{2}+1}t^{\frac{1}{2}+1} + C$$

$$= \pi\frac{1}{\frac{3}{2}}t^{\frac{3}{2}} + C$$

$$= \pi\frac{2}{3}t^{\frac{3}{2}} + C$$

$$= \frac{2\pi}{3}t^{3/2} + C.$$ ✓

29. $\int(1+x+e^x)\,dx = \int 1\,dx + \int x\,dx + \int e^x\,dx$

$= \int dx + \int x\,dx + \int e^x\,dx$

$= x + \frac{1}{2}x^2 + e^x + C$

35. $\int\left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)dx = \int\sqrt{x}\,dx + \int\frac{3}{\sqrt{x}}\,dx$

$= \int x^{\frac{1}{2}}\,dx + \int 3x^{-1/2}\,dx$

$= \int x^{\frac{1}{2}}\,dx + 3\int x^{-\frac{1}{2}}\,dx$

$= \frac{1}{\frac{1}{2}+1}\,x^{\frac{1}{2}+1} + 3\left(\frac{1}{-\frac{1}{2}+1}\,x^{-\frac{1}{2}+1}\right) + C$

$= \frac{1}{\frac{3}{2}}\,x^{\frac{3}{2}} + 3\left(\frac{1}{\frac{1}{2}}\,x^{\frac{1}{2}}\right) + C$

$= \frac{2}{3}x^{3/2} + 3\left(2x^{\frac{1}{2}}\right) + C$

$= \frac{2}{3}x^{3/2} + 6x^{1/2} + C$

$= \frac{2}{3}x^{3/2} + 6\sqrt{x} + C.$ ✓

37. $\displaystyle\int\left(\frac{u^3+2u^2-u}{3u}\right)du$

$\displaystyle=\int\left(\frac{u^3}{3u}+\frac{2u^2}{3u}-\frac{u}{3u}\right)du$

$\displaystyle=\int\left(\frac{1}{3}u^2+\frac{2}{3}u-\frac{1}{3}\right)du.$

$\displaystyle=\int\frac{1}{3}u^2\,du+\int\frac{2}{3}u\,du-\int\frac{1}{3}\,du$

$\displaystyle=\frac{1}{3}\int u^2\,du+\frac{2}{3}\int u\,du-\frac{1}{3}\int du$

$\displaystyle=\frac{1}{3}\left[\frac{1}{3}u^3\right]+\frac{\cancel{2}}{3}\left[\frac{1}{\cancel{2}}u^2\right]-\frac{1}{3}\left[u\right]+C$

$\displaystyle=\frac{1}{9}u^3+\frac{1}{3}u^2-\frac{1}{3}u+C.$

38. $\displaystyle\int\frac{x^4-1}{x^2}\,dx=\int\left(\frac{x^4}{x^2}-\frac{1}{x^2}\right)dx$

$\displaystyle=\int\left(x^2-\frac{1}{x^2}\right)dx$

$\displaystyle=\int x^2\,dx-\int\frac{1}{x^2}\,dx$

$\displaystyle=\int x^2\,dx-\int x^{-2}\,dx$

$\displaystyle=\frac{1}{3}x^3-\frac{1}{-2+1}x^{-2+1}+C$

$\displaystyle=\frac{1}{3}x^3+x^{-1}+C$

$\displaystyle=\frac{1}{3}x^3+\frac{1}{x}+C.$

45. $\int (e^t + t^e)\,dt$

$= \int e^t\,dt + \int t^e\,dt$

$= e^t + \frac{1}{e+1} t^{e+1} + C$ ✓

48. $\int \frac{t^3 + \sqrt[3]{t}}{t^2}\,dt = \int \frac{t^3 + t^{1/3}}{t^2}\,dt$

$= \int \left(\frac{t^3}{t^2} + \frac{t^{1/3}}{t^2} \right) dt$

$= \int \left(t + t^{-5/3} \right) dt$

$\frac{1}{3} - 2 = \frac{1}{3} - \frac{6}{3} = -\frac{5}{3}$

$= \int t\,dt + \int t^{-5/3}\,dt$

$= \frac{1}{2} t^2 + \frac{1}{-\frac{5}{3}+1} t^{-5/3+1} + C$

$= \frac{1}{2} t^2 + \frac{1}{-\frac{2}{3}} t^{-\frac{2}{3}} + C$

$= \frac{1}{2} t^2 - \frac{3}{2} t^{-3/2} + C.$

$= \frac{1}{2} t^2 - \frac{3}{2t^{3/2}} + C$ ✓

49. Simplify the integrand first.

$$\frac{(\sqrt{x}-1)^2}{x^2} = \frac{(\sqrt{x}-1)(\sqrt{x}-1)}{x^2}$$

$$= \frac{x - \sqrt{x} - \sqrt{x} + 1}{x^2}$$

$$= \frac{x - 2\sqrt{x} + 1}{x^2}$$

$$= \frac{x}{x^2} - \frac{2\sqrt{x}}{x^2} + \frac{1}{x^2}$$

$$= \frac{1}{x} - 2x^{-3/2} + x^{-2}$$

$$= x^{-1} - 2x^{-3/2} + x^{-2}$$

Then

$$\int \frac{(\sqrt{x}-1)^2}{x^2}\,dx = \int (x^{-1} - 2x^{-3/2} + x^{-2})\,dx$$

$$= \int x^{-1}\,dx - 2\int x^{-3/2}\,dx + \int x^{-2}\,dx$$

$$= \ln|x| - \frac{2}{-\frac{3}{2}+1}\,x^{-\frac{3}{2}+1} + \frac{1}{-2+1}\,x^{-2+1} + C.$$

$$= \ln|x| + \frac{2}{\frac{1}{2}}\,x^{-\frac{1}{2}} - x^{-1} + C$$

$$= \ln|x| + \frac{4}{\sqrt{x}} - \frac{1}{x} + C.$$ ✓

Remember that $\int \frac{1}{x}\,dx = \ln|x| + C$