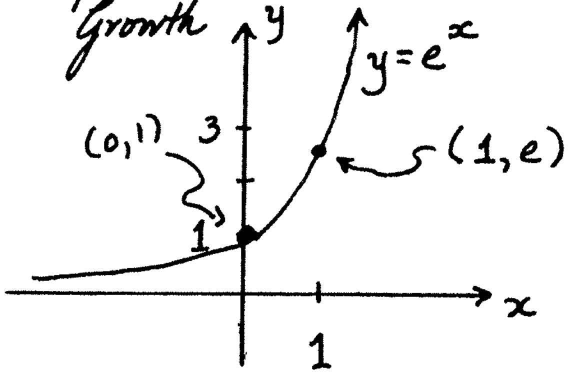
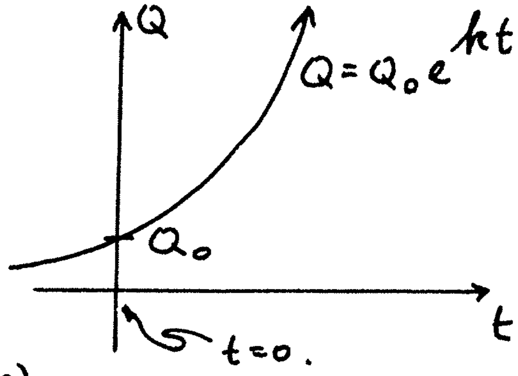
§5.6 – Exponential Functions as Mathematical Models.

Exponential Growth $\quad Q(t) = Q_0 e^{kt} \qquad t \in [0, \infty)$





$$
\begin{aligned}
&Q_0 e^{k(0)}_0 \\
&= Q_0 \cancel{e^0}_1 \\
&= Q_0
\end{aligned}
\qquad \therefore (0, Q_0)
$$

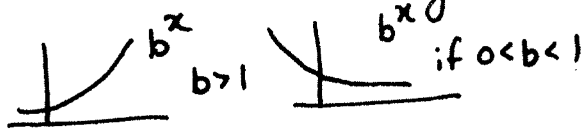
Q_0 is the initial amount $\qquad Q(0) = Q_0$.

k is called the growth constant.

$$\frac{d}{dx} e^x = e^x \left\{ \frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x) \right.$$

$$
\begin{aligned}
Q(t) &= Q_0 e^{kt} \\
Q'(t) &= \frac{d}{dt}(Q_0 e^{kt}) \\
&= Q_0 \frac{d}{dt}(e^{kt}) \\
&= Q_0 k e^{kt} \\
&= k \underbrace{Q_0 e^{kt}} \\
&= k\, Q(t)
\end{aligned}
\qquad
\begin{aligned}
\frac{d}{dt} e^{kt} &= e^{kt} \cdot k \\
&= k e^{kt}
\end{aligned}
$$

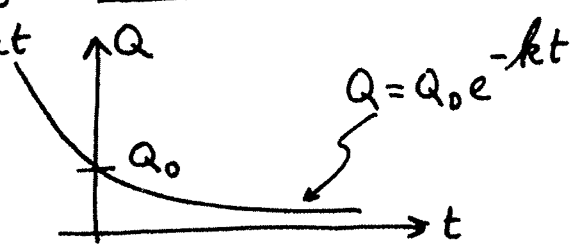
$k > 0 \Rightarrow$ increasing.



Exponential Decay. ~~If k<0~~

$$
\begin{aligned}
Q(t) &= Q_0 e^{-kt} \\
&= Q_0 \left(\tfrac{1}{e}\right)^{kt}
\end{aligned}
$$

$0 < \frac{1}{e} < 1$



Example Fort A

$$Q(t) = \frac{5000}{1 + 1249e^{-kt}}$$

Q # of soldiers who have flu
t days

If 40 soldiers that have the flu by $t = 7$ (7 days), how many will have it on $t = 15$.

Solution $Q(7) = 40$

$$\frac{5000}{1 + 1249e^{-7k}} = 40 \qquad \text{Solve for } k.$$

$$5000 = 40(1 + 1249e^{-7k})$$

$$\frac{5000}{40} = 1 + 1249e^{-7k}$$

$$125 = 1 + 1249e^{-7k}$$

$$125 - 1 = 1249e^{-7k}$$

$$\frac{124}{1249} = \frac{1249}{1249}e^{-7k}$$

$$\frac{124}{1249} = e^{-7k}$$

$$\ln\left(\frac{124}{1249}\right) = \ln e^{-7k}$$

$$\ln\left(\frac{124}{1249}\right) = -7k \underbrace{\ln e}_{1}$$

$$\ln\left(\frac{124}{1249}\right) = -7k$$

$$-\frac{\ln(124) - \ln(1249)}{7} = k$$

$$k \approx 0.33$$

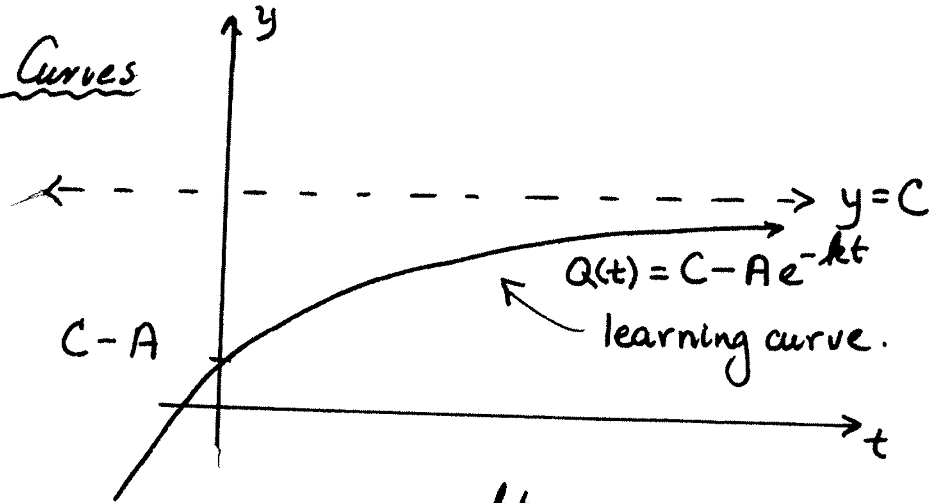
$$Q(t) = \frac{5000}{1 + 1249e^{-0.33t}}$$

$$Q(15) = \frac{5000}{1+1249e^{-0.33(15)}}$$

$$\approx \cancel{507.75}\ 508.$$

$\therefore$ approx 508 soldiers are infected by the 15^{th} day.

Learning Curves



$$Q'(t) = -Ae^{-kt}(-k)$$

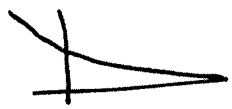
$$= Ake^{-kt} > 0 \text{ for all } t \in [0, \infty).$$

horizontal asymptote

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} \left(C - Ae^{-kt}\right)$$

$$= \underbrace{\lim_{t\to\infty} C}_{} - \lim_{t\to\infty} Ae^{-kt}$$

$$= C - A\underbrace{\lim_{t\to\infty} e^{-kt}}_{0}$$



$$= C - A(0)$$

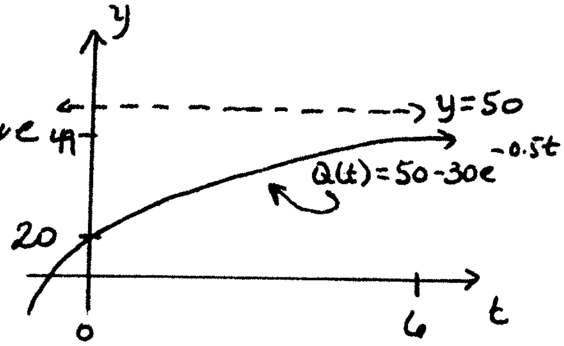
$$= C$$

Example Basic training program.

Laptop.

↳ worker follows the learning curve

$$Q(t) = 50 - 30e^{-0.5t}$$



# of laptops per day.

t months after worker begins employment.

a) How many laptops can a new employee assemble per day at the beginning.

$t=0$.

$$Q(0) = \overset{C}{50} - \overset{A}{30}e^{-0.5(0)}$$
$$= 50 - 30\underbrace{e^{0}}_{1}$$
$$= 50 - 30(1)$$
$$= 50 - 30$$
$$= 20 \text{ laptops in a day.}$$

b) after 1 month of experience?

6 months?

$$Q(1) = 50 - 30e^{-0.5(1)}$$
$$= 50 - 30\underbrace{e^{-0.5}}_{0.6065}$$
$$= 50 - 18.1959$$
$$= 31.8041 \approx 32 \text{ laptops.}$$

$$Q(6) = 50 - 30e^{-0.5(6)}$$
$$= 50 - 30e^{-3}$$
$$= 50 - 30(0.0498)$$
$$= 50 - 1.4936$$
$$= 48.5064 \approx 49 \text{ laptops.}$$

$Q_0 e^{kt}$

1. $Q(t) = 400e^{0.05t}$ $\quad$ t mins

Q # of bacteria.

a) $k = 0.05$ $\quad$ What is the growth constant?

b) What quantity was initially present?

$$Q(0) = 400\underbrace{e^{0.05(0)}}_{e^0 = 1}$$

$t=0$ $\quad$ $= 400.$

400 initially present

c)

t	0	10	20	100	1000
$Q(t)$