Hemodynamics:
A Source of Intrigue for Pure and Applied Mathematicians

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Outline

Introduction

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- French physician and physiologist
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- He was interested in flow of human blood in narrow tubes

Poiseuille's Law

Theorem (Poiseuille's Law)

The volume flux Q for a cylinder with length L is given by

$$Q = \frac{\pi}{8} \frac{p_1 - p_2}{\mu L} a^4$$

where p_1 and p_2 denotes the pressure at the beginning and the end of the tube respectively, and a is the radius.

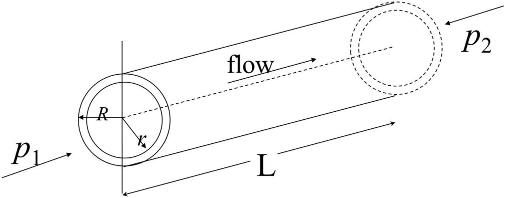

Figure: This is a cylinder.

Properties of Poiseuille's Law

Corollary (Properties of Poiseuille's Law)

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3. *The flux is inversely proportional to the length of the cylinder L.*

Navier-Stokes equation

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Definition (Viscosity)

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- Assuming constant viscosity μ then, the NS equation is

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}\vec{u}\right)=-\nabla p+\mu\vec{\nabla}^2\vec{u}+\vec{f} \quad (1)$$

where ρ is the density, p is the pressure, $\vec{f}$ is the body forces, and $\vec{u}=(u,v,w)$ is the flow velocity with components u, v, and w.

Derivation of Poiseuille's Law

We will derive Poiseuille's Law in this section using the NS equation for an incompressible fluid.
With the axisymmetric assumption that $u = u(x, r)$, the NS equation reduces to

$$0 = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$
$$\frac{\partial p}{\partial x} = \mu \nabla^2 u$$

where ∇^2 in cylindrical coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}$$

Derivation of Poiseuille's Law

From the r component of the NS equation,

$$\rho\left(\frac{\partial}{\partial t}+(\vec{u}\cdot\nabla)\right)u_r=-(\nabla t)_r+\mu\nabla^2 u_r$$

and u is the only nonzero velocity component, which means that $\frac{\partial p}{\partial r}=0 \implies$ pressure is only dependent on x. *i.e.*, $p=p(x)$.

Derivation of Poiseuille's Law

Since $u = u(r)$ and $p = p(x)$, by a separation of variables argument, we conclude each term of the equation is constant. Thus,

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \text{constant.}$$

This means that the pressure is simply a linear function

$$p = p_1 + \frac{p_2 - p_1}{L} x$$

where p_1 is the initial pressure at $x = 0$, L is the "slope", and p_2 is the pressure at $x = L$.

Derivation of Poiseuille's Law

Then

$$\frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = \frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = \frac{r}{\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)$$

Integrating with respect to r yields

$$\int \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)\,\mathrm{d}r = \int \frac{r}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\,\mathrm{d}r$$

$$r\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{r^2}{2} + C$$

where C is an arbitrary constant.

Derivation of Poiseuille's Law

$$r\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{r^2}{2} + C$$

Dividing by r

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\frac{r}{2} + \frac{C}{r}$$

and integrating with respect to r

$$u = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{r^2}{4} + D$$

where D is an arbitrary constant.

Derivation of Poiseuille's Law

No-slip condition: $u = 0$ at $r = a$.
We can then find D

$$D = -\frac{1}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}x}a^2$$

Substituting D into u yields,

$$u = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}r^2 - \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}a^2 \quad (2)$$

$$= \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}[r^2 - a^2] \quad (3)$$

We can use Equation (3) that we derived for u to find the volume flux for the length of the cylinder L.

Derivation of Poiseuille's Law

$$
\begin{aligned}
Q &= \int_0^a 2\pi r u \, \mathrm{d}u \\
&= \int_0^a 2\pi r \left[\frac{1}{4\mu} \left(\frac{\mathrm{d}p}{\mathrm{d}x} \right) (r^2 - a^2) \right] \mathrm{d}r \\
&= \int_0^a 2\pi r \left[\left(\frac{p_2 - p_1}{4\mu L} \right) (r^2 - a^2) \right] \mathrm{d}r \\
&= \int_0^a 2\pi r^3 \left(\frac{p_2 - p_1}{4\mu L} \right) \mathrm{d}r - \int_0^a 2\pi r \left(\frac{p_2 - p_1}{4\mu L} \right) a^2 \mathrm{d}r \\
&= \frac{2\pi r^4}{4} \left(\frac{p_2 - p_1}{4\mu L} \right) \Bigg|_0^a - \frac{2\pi r^4}{2} \left(\frac{p_2 - p_1}{4\mu L} \right) a^2 \Bigg|_0^a \\
&= \frac{\pi a^4}{8} \cdot \frac{p_2 - p_1}{\mu L} - \frac{2\pi a^4}{8} \cdot \frac{p_2 - p_1}{\mu L} \\
&= -\frac{\pi a^4}{8} \cdot \frac{p_2 - p_1}{\mu L} \\
&= \frac{\pi}{8} \frac{p_1 - p_2}{\mu L} a^4
\end{aligned}
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- For arteries, when they become restricted, then the blood pressure in the artery required to supply the blood will rise considerably to the fourth power of the artery's radial reduction $\rightarrow$ hypertension

Kilner *et al.* (1993)

- Studied the aortic arch using 3D MRI velocity mapping

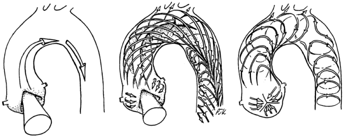

Figure: Schematic drawings to illustrate typical aortic arch flow development. From Kilner *et al.* (1993).

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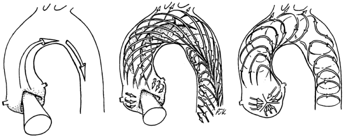

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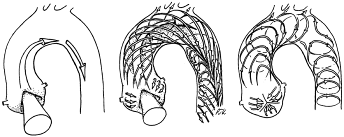

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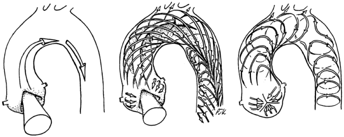
- The continuous flow gave rise to a right-handed helical flow field in the upper curvature of the anatomically twisted arch, and during pulsatile flow, a right-handed helix came to dominate during the systole.
- The right-handed helical flow dominated when the walls of the aorta traced part of a right-handed helical path.



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- With the 3D aorta geometry, it was imported into Autodesk Multiphysics Simulation 2012 to run the fluid simulation

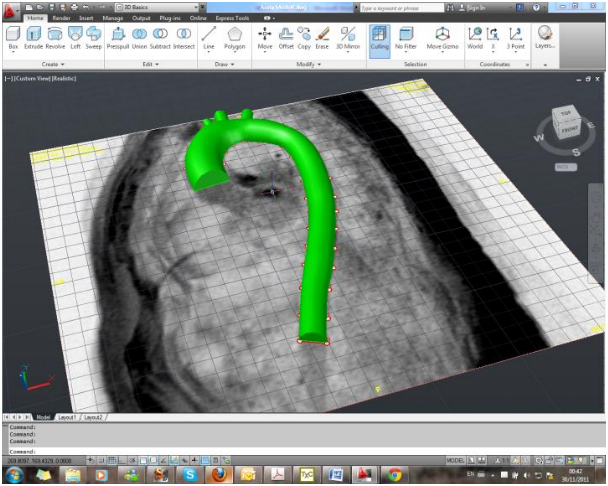

Figure: Creating the aortic geometry in Autodesk AutoCAD 2012

Numerical Simulation

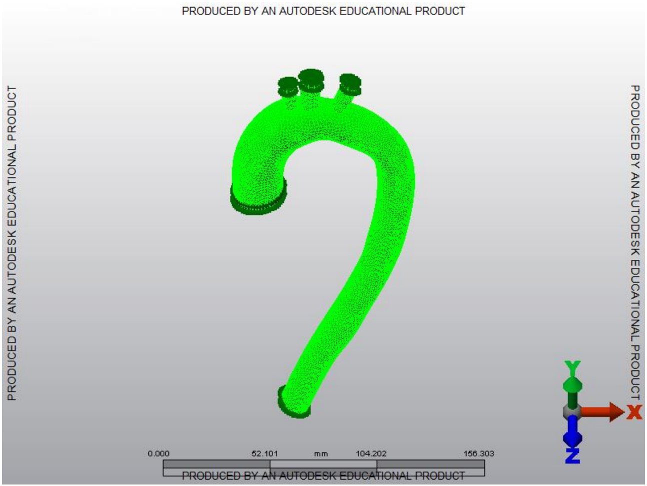

Figure: Meshed aortic geometry in Autodesk Multiphysics Simulation 2012

Numerical Simulation

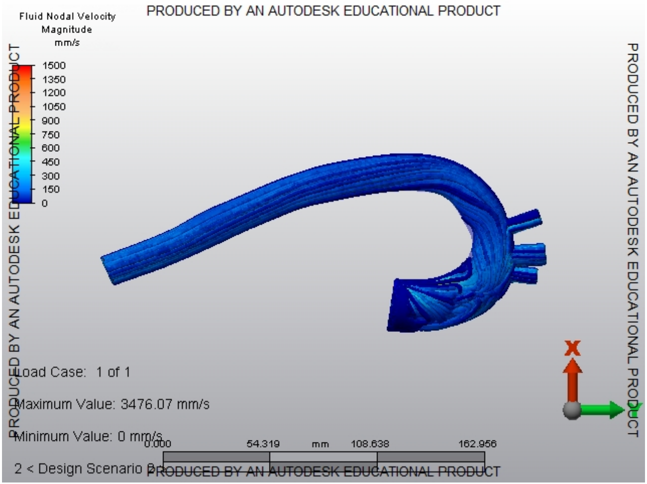

Figure: Steady state velocity streamline in the aorta.

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- Found a strong inverse relationship between the oscillating shear index and their helical flow index.
- Suggested that the presence of helical flow may play a significant role in preventing atherogenesis.

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Questions for the Future

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2. What is the target amount of helicity in the aorta?
3. How do we measure it non-invasively?

Bibliography

Kilner, P. et al. Helical and retrograde secondary flow patterns in the aortic arch studied by three-directional magnetic resonance velocity mapping. Circulation 88, 2235–2247.

Morbiducci, U., et al. Helical flow as fluid dynamic signature for atherogenesis risk in aortocoronary bypass: A numeric study. Journal of Biomechanics 40, 519–534.